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Attractors and recurrence for dendrite-critical polynomials ☆

Alexander Blokh [a,*], Michał Misiurewicz [b]

[a] *Department of Mathematics, University of Alabama in Birmingham, University Station, Birmingham, AL 35294-2060, USA*

[b] *Department of Mathematical Sciences, IUPUI, 402 N. Blackford Street, Indianapolis, IN 46202-3216, USA*

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Abstract

We call a rational map f *dendrite-critical* if all its recurrent critical points either belong to an invariant dendrite D or have minimal limit sets. We prove that if f is a dendrite-critical polynomial, then for any conformal measure μ either for almost every point its limit set coincides with the Julia set of f, or for almost every point its limit set coincides with the limit set of a critical point c of f. Moreover, if μ is non-atomic, then c can be chosen to be recurrent. A corollary is that for a dendrite-critical polynomial and a non-atomic conformal measure the limit set of almost every point contains a critical point.

Keywords: Complex dynamics; Attractors; Conformal measures; Postcritical set

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* Corresponding author.

E-mail addresses: ablokh@math.uab.edu (A. Blokh), mmisiure@math.iupui.edu (M. Misiurewicz).

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1. Introduction

The central question in the Dynamical Systems Theory is that of the long term behavior of orbits. To address this question, one often studies ω-limit sets for orbits typical in some sense. In this paper we do it for a class of complex polynomials, understanding "typical" in terms of conformal measures. The paper continues our previous paper on a similar topic [10] and aims at describing ω-limit sets of points which are realized on sets of positive conformal measure μ. However, in addition to this question, which essentially dates back to Milnor [21], we are also interested in the following related problem: is it true that for μ-a.e. point x the ω-limit set $\omega(x)$ contains a critical point?

To fix terminology and notation, recall that for a continuous map T of a compact Hausdorff space X to itself and a point $x \in X$ the *orbit of* x is the sequence $(f^n(x))_{n=0}^{\infty}$ (we denote it $\mathrm{orb}(x)$ and sometimes consider it to be a set rather than a sequence), and the *ω-limit set* of x is the set of all accumulation points of $\mathrm{orb}(x)$. We denote the latter $\omega(x)$ and usually call it simply the *limit set of* x.

Let us describe ideas motivating our research. Milnor in [21] introduced the notion of an *attractor*, and, in particular, *primitive attractor* (for a given measure μ a *primitive attractor* is a set A such that $\mu(\{x\colon\ A = \omega(x)\}) > 0$). Milnor [21] conjectured that in "good" cases (i.e., for "good" maps and measures) there are finitely many primitive attractors and an a.e. point in the sense of the measure has the limit set coinciding with one of the primitive attractors.

In some cases Milnor's conjecture was verified. In all such cases the following is shown. Given a "good" map f there is a finite set of points C_f such that for any "good" measure μ at least one of the following holds:

(1) the map f is ergodic with respect to μ, the support of μ coincides with a special set $A(f)$ (usually $A(f)$ is the non-wandering set of f or a version of it—e.g., in the complex case it is the Julia set of f), and for μ-a.e. x the set $\omega(x)$ coincides with the support of μ;
(2) for μ-a.e. x there exists $c(x) \in C_f$ such that $\omega(x) = \omega(c(x))$.

In the future if this takes place we say that the *Milnor decomposition* (of $A(f)$) holds. Moreover, if C_f is the set of all critical points of f we say that the *critical Milnor decomposition* (of $A(f)$) holds. The main results in this field establish the (critical) Milnor decomposition for various classes of maps and measures. Observe that while in the case of smooth interval maps the Lebesgue measure on the interval seems to be a natural choice for μ, in the case of rational maps the natural choice for μ is any conformal measure.

The most thoroughly studied case here is that of smooth interval maps with Lebesgue measure for which the Milnor decomposition is essentially established in [4–7,15] (see also [9]). In the case of the Julia set of a rational complex map with a conformal measure much less has been done. Indeed, working with these maps is more complicated, because the space then is two-dimensional. This allows a map to have much more flexibility in terms of its dynamics, which is not always compensated by nice analytic properties of the map. Still, some results in this direction have been obtained; to state them we need the following definitions. Given a rational map f a measure μ on $J(f)$ is *conformal* (for f)

if for an *exponent* $\alpha > 0$ we have $\mu(f(A)) = \int_A |f'(z)|^\alpha \, d\mu$ whenever $f|_A$ is 1-to-1 (by [25] f has at least one conformal measure). Also, a point x is said to be *precritical* if it eventually maps into a critical point; x is said to be *preparabolic* if it eventually maps into a parabolic periodic point (whose orbit by the Fatou theorem is the limit set of some critical point).

The following theorem has been proven in [8] (cf. [14]). Denote by $P_r(f)$ the union of the limit sets of recurrent critical points of f.

Theorem 1.1. *At least one of the following holds for a conformal measure μ of a rational map f:*

(1) *For μ-a.e. point x we have $\omega(x) = J(f)$.*
(2) *For μ-a.e. point x at least one the following holds*:
 (a) *$\omega(x) \subset P_r(f)$, or*
 (b) *x is a precritical, or*
 (c) *x is preparabolic.*

The aim of Theorem 1.1 is to deal with conformal measures with *no assumptions* on rational maps f. Observe, that even though the conclusions of Theorem 1.1(1) are quite strong and give the description of primitive attractors for the measure μ (in that case the only primitive attractor is $J(f)$), the conclusions of Theorem 1.1(2) are weaker and do not provide such description. Thus in general it is not known if Milnor decomposition of $J(f)$ holds in general for rational maps f; to establish it one needs appropriate assumptions on the map.

There are two types of assumptions considered in the literature in this context. First of all, these are conditions of analytic nature which single out maps with so-called *non-uniform hyperbolicity* (e.g., Collet–Eckmann conditions or topological Collet–Eckmann conditions). A lot of deep results concerning non-uniformly hyperbolic rational maps can be found in literature (see, e.g. [13,22] or [24]). These results easily imply the critical Milnor decomposition of $J(f)$ for non-uniformly hyperbolic rational maps.

However, it turns out that rather strong assumptions which define non-uniformly hyperbolic rational maps are not necessary for the existence of the Milnor decomposition of the Julia set. There are other principally different types of assumptions on the maps which imply the same conclusion. To begin with, these are topological assumptions on the Julia set of a map, and indeed the corresponding results were obtained in [11] (see also [1–3,17]). Since we study conformal measures which are all supported on the Julia sets, it is no wonder that the topology of the Julia sets is crucial here. However, it has been recently discovered in [10] that in some cases it is the *topological structure of the orbits and limit sets of critical points* which determines if the critical Milnor decomposition of $J(f)$ holds.

In the present paper we improve the results of [10] and suggest another and in a sense more general set of assumptions on a map which still allow us to conclude that the critical Milnor decomposition of the Julia set holds. In addition to that, we deduce several important extra properties of the limit behavior of typical points in the sense of a conformal measure. Our aim is to discover true topological causes of the critical Milnor decomposition for complex maps, and we are motivated by our belief that these must be related to the limit behavior of critical points.

The tools employed here combine both analytical and topological approaches. For example, an important technical result of this paper is an analytic in its nature Theorem 3.5 which uses the notion of *recurrent criticality* introduced in Section 3 and extends results of Mañé ([19], see also [23]). Together with the topological analysis of dynamics on dendrites made in Section 4, Theorem 3.5 allows us to obtain the main results of the paper.

We would like now to state the results of [10,11] (see also [1–3,17]). To do so we need the following definitions. A set A is said to be *minimal* if the map restricted to this set is minimal (i.e., the orbit of every point of A is dense in A). A *graph* is a one-dimensional branched manifold. Now, we consider in [10] rational maps for which each critical point either belongs to an invariant graph G, or has minimal limit set, or is non-recurrent and has the limit set disjoint from G. We call such maps *graph-critical*. Let us point out that G above is just a topological graph, and hence it is unknown whether graph-critical polynomials have locally connected Julia sets.

The following theorem combines the results of [10] and [11] (see also [1–3,17]).

Theorem 1.2. *Suppose that f is a polynomial with locally connected Julia set or f is a graph-critical rational map. Then the critical Milnor decomposition of $J(f)$ holds for any conformal measure.*

In this paper we aim at finding assumptions of topological nature which are weaker than graph criticality but would still yield critical Milnor decomposition of $J(f)$. It turns out that this can be done for polynomials. Let us introduce necessary terminology. A continuum is said to be a *dendrite* if it is locally connected and tree-like (contains no simple closed curves). A rational map f is *dendrite-critical* if there exists a (perhaps empty) forward invariant dendrite D such that every recurrent critical point of f either belongs to D, or has minimal limit set. In particular, if all recurrent critical points have minimal limit sets then such rational map is dendrite-critical. We do not make any assumptions as to whether D is contained in $J(f)$ or not. We consider dendrite-critical *polynomials*; thus, we study narrower class than all rational maps, but we make weaker topological assumptions and also restrict only the behavior of their *recurrent* critical points.

The following theorem shows that the critical Milnor decomposition holds for dendrite-critical polynomials and their conformal measures.

Theorem 5.4. *For a dendrite-critical polynomial f and a conformal measure μ at least one of the following holds*:

(1) *For μ-a.e. $x \in J(f)$, $\omega(x) = J(f)$.*
(2) *For μ-a.e. $x \in J(f)$, $\omega(x) = \omega(c(x))$ for some critical point $c(x)$ depending on x, and at least one of the following holds*:
   (a) *x is an eventual preimage of $c(x)$, or*
   (b) *x is preparabolic, or*
   (c) *$c(x)$ can be chosen to be recurrent.*

The following is the main corollary of Theorem 5.4.

Corollary 5.5. *If f is a dendrite-critical polynomial and μ is a non-atomic conformal measure, then for μ-a.e. point x the set $\omega(x)$ contains a critical point.*

We would like to point out that a result similar to Corollary 5.5 for smooth interval maps is the main result of [18]. It serves as a basic ingredient of the results of [15], in particular for the construction of ergodic decomposition there. We hope that in the context of dendrite-critical polynomials Corollary 5.5 may serve the same purpose.

The paper is arranged as follows. In Section 2 we go over a dynamical construction from [10] which allows one to make conclusions like the ones of Theorem 1.2 in the general setting of continuous maps on a compact metric space. In Section 3 we introduce the notion of recurrent criticality and extend some results of Mañé for this notion. The main result of Section 3 is Theorem 3.5 which could prove to be important for applications. Section 4 is devoted to a detailed study of dynamics on dendrites under rational and polynomial maps. Finally, we prove the main results of the paper in Section 5.

2. Basic facts about followed points

In this section we list those results of [10, Section 3] needed in what follows. Throughout the rest of this section $T : X \to X$ is a continuous map of a metric compact space X with metric d and $C \subset X$ is a finite set. In this not necessarily smooth situation one can still call a periodic point a of period m *repelling* (*topologically*) if in some metric d_1 equivalent to d, for some $\varepsilon > 0$ and any point $x \neq a$ which is at most ε away from a, we have $d_1(T^m(x), a) > d_1(x, a)$. If we use the fact that some point is repelling, we will assume that our metric is already modified as above.

Now we introduce our basic setup. It consists of definitions and notation, and depends on a choice of a point $x \in X$ and a set C. We give it a special name since we will have to refer to it several times.

Basic Setup. Suppose that $x \in X$ and for every integer $i \geqslant 0$ an integer $m_i \in [0, i]$ and a point $c_i \in C$ are chosen. Then we use the following definitions and notation:

(1) For a given $c \in C$ with infinite sequence of numbers m_i with $c_i = c$, if the sequence does not tend to ∞ then we call this case *bounded* (for c), otherwise we call the case *unbounded* (for c).
(2) A pair of points $(T^r(x), T^{r-m_i}(c_i))$, $m_i \leqslant r \leqslant i$, is called an *$i$-pair*.
(3) For a given $c \in C$ with infinite sequence of numbers m_i such that $c_i = c$, the set of all accumulation points of the sequence $(T^{m_i}(x))$ will be denoted by L_c. Clearly, $L_c \subset \overline{\mathrm{orb}(x)}$. Moreover, in the unbounded case for c we have $L_c \subset \omega(x)$.
(4) A pair of points (x', c') which is the limit for some sequence of i-pairs with $i \to \infty$ and $c_i = c \in C$ with $\omega(c)$ not minimal, is called a *limiting pair*.

The following condition was called basic in [10].

Basic Condition. We have $d(T^i(x), T^{i-m_i}(c_i)) \to 0$ as $i \to \infty$.

With the Basic Setup, we will say that x is *C-followed* if Basic Condition holds and for any limiting pair (x', c') we have $\omega(x') = \omega(c')$. The simplest Basic Setup is Standard Basic Setup described in Introduction for a persistent point x of a rational map f. Then for any i we have a generating pair (c_i, m_i) for r_i and by the definition of a persistent point Basic Condition is satisfied.

The main general result of [10] is that if a point is C-followed, then its limit set coincides with the limit set of one of the points of C. Here we provide a bit more detailed statement than the one literally given in [10]; the main technical addition is that we emphasize that the point c whose limit set coincides with that of x can be chosen so that it appears infinitely many times in Basic Setup.

Theorem 2.1. *If x is C-followed, then $\omega(x) = \omega(c)$ for some $c \in C$. Moreover, this c can be chosen in such a way that c appears in Basic Setup infinitely many times and if the bounded case holds for it then $T^m(x) \in L_c$ for some $m \in \mathbb{N}$, while in the unbounded case $L_c \subset \omega(x)$, each point of L_c is recurrent and has limit set coinciding with $\omega(x)$.*

There is a certain standard way to construct Basic Setup for rational maps (we discuss it in Section 5). This approach is quite fruitful in some cases, e.g. it leads to the proof of Theorem 1.2 in the case when the Julia set is locally connected, see [11] and also [1–3,17]. However, in the case of a general rational map f this Standard Basic Setup does not satisfy conditions necessary for the point to be C-followed (here C is the set of critical points of f), because in general for limiting pairs (x', y') we do not know if $\omega(x') = \omega(y')$. The idea of [10] was to suggest a different Basic Setup which uses topological and dynamical properties of graph maps and thus shows that persistent points are C-followed. However, the Basic Setup from [10] does not imply that c in Theorem 2.1 can be chosen so that $c \in \omega(x)$; to some extent the aim of this paper is to develop tools which would allow to make such conclusion in some cases.

3. A version of the results of Mañé relative to pull-backs

In this section a version of the results from [19] is obtained. The main ideas are from [19], the difference being that instead of considering points not belonging to the limit sets of recurrent critical points we consider all points, but at the same time work only with the pull-backs of their neighborhoods which do not contain recurrent critical points. The proofs are close to those by Mañé, but for the sake of completeness and also because the conditions are different we include full proofs in this section (except Lemma 3.1). To state the results we need to introduce the notation which mimics that of Mañé. Let us point out that even though the main results of the paper deal with polynomials, in this section we work with a rational map f. Also, we would like to point out that we use the same geometrical construction as Mañé in his classical paper, thus we use squares rather than disks.

We need some terminology. If $\hat{\mathbb{C}}$ is the closed complex plane and $f : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ is a rational map, we denote by $C(f) = C$ the set of its critical points. For a point x let $B(x, r)$ be the open disk of radius r centered at x. A *Jordan disk* is a set U, homeomorphic to an open disk with $\bar{U}$ homeomorphic to a closed disk such that U is the interior of $\bar{U}$; *closed Jordan disks* are closures of the open ones.

Suppose that A is a connected set. Then a component V of $f^{-n}(A)$ is said to be an *(n)-pull-back* of A, and sets $f(V), f^2(V), \ldots, f^{n-1}(V)$ are called *pull-backs of A corresponding to V*. If f is univalent on all $V_i = f^i(V)$, $i < n$, then we say that this pull-back is *univalent*. Suppose that a connected set $B \subset A$ (or $B \supset A$) is given. Then any k-pull-back of B contained in a k-pull-back of A corresponding to V (or containing a k-pull-back of A corresponding to V) is also said to be *corresponding* to V. We will mostly deal with these notions when A or B are Jordan disks, but the definitions can be given in general.

By a *square* we understand a square whose boundary segments are vertical and horizontal. The half-length of its side is called the *radius* of the square, and the point of intersection of its diagonals is said to be its *center*. Given a square of radius δ centered at p, denote by S^k the square of radius $k\delta$ centered at p. Also, suppose that U is a Jordan disk and V is a pull-back of U, i.e., V is a component of $f^{-n}(U)$ for some n. Then the number of critical points hit by the sets $V, f(V), \ldots, f^{n-1}(V)$ is said to be the *criticality* of $f^n|_V$ (or the *criticality* of V if n is fixed). On the other hand, the number of *recurrent* critical points hit by the sets $V, f(V), \ldots, f^{n-1}(V)$ is said to be the *recurrent criticality* of $f^n|_V$ (or the *recurrent criticality* of V if n is fixed). Finally, V is said to be a *non-recurrent* pull-back of U if $f(V), \ldots, f^n(V) = U$ do not contain recurrent critical points.

Note that in this section when a pair of squares with the same center appears, sometimes they are denoted by S and S^k with $0 < k < 1$, and sometimes by S^l and S with $l > 1$, depending on whether the larger or the smaller square was the original one.

We arrange the section as follows: first we establish a sequence of technical but useful lemmas and then prove the main result. First we include (without proof) Lemma 3.1 proven in [19] (see also [23]).

Lemma 3.1. *Suppose that numbers $\varepsilon > 0$, $0 < k < 1$, $\gamma > 0$ and N are given. Then there exists $\delta = \delta(\varepsilon, k, \gamma, N)$ such that the following holds. Let S be a square of radius less than δ such that $d(S, p) > \gamma$ for any parabolic or attracting periodic point p. Suppose that V is an n-pull-back of S such that $f^n|_V$ has criticality at most N. Then any n-pull-back of S^k corresponding to V has diameter at most ε.*

Essentially, the aim of this section is to prove that in Lemma 3.1 one can replace the assumption about the criticality of V by the corresponding assumption about the recurrent criticality of V. To do so we establish some other technical lemmas and introduce certain constants. First of all, let us denote by ξ a positive number such that any non-recurrent critical point c of f never comes closer than ξ to itself. This implies the following lemma.

Lemma 3.2. *Let A be a set such that $A, f(A), \ldots, f^{n-1}(A)$ are sets of diameter less than ξ. Then every non-recurrent critical point of f is covered by at most one set $f^i(A)$, $0 \leqslant i \leqslant n-1$. Also, if A is a pull-back of a Jordan disk $f^n(A)$ such that recurrent criticality of $f^n|_A$ is r, then the criticality of $f^n|_A$ is at most $d + r$.*

Proof. Assume that c is a non-recurrent critical point such that $c \in f^i(A) \cap f^j(A)$ and $i < j$. Then $f^{j-i}(c) \in f^j(A)$ and so we have $d(f^{j-i}(c), c) < \xi$ which contradicts the choice of ξ. The second part of the lemma follows immediately. $\square$

The next lemma is similar to Lemma 3.1, but is sometimes more convenient for our purposes.

Lemma 3.3. *Suppose that numbers $\varepsilon > 0$, $0 < k < 1$, $\gamma > 0$ and r are given. Let $\delta = \delta(\varepsilon, k, \gamma, r + d)$ be the number found in Lemma* 3.1. *Let S be a square of radius less than δ such that $d(S, p) > \gamma$ for any parabolic or attracting periodic point p, let V be an n-pull-back of S and V' be a corresponding to V n-pull-back of S^k. Suppose that* $\operatorname{diam}(V') \geqslant \varepsilon$ *and the recurrent criticality of $f^n|_V$ is at most r. Then there exists i, $0 \leqslant i \leqslant n - 1$ such that* $\operatorname{diam}(f^i(V)) \geqslant \xi$.

Proof. Let us assume that the number i with required properties does not exist. Then for any i, $0 \leqslant i \leqslant n - 1$ we have $\operatorname{diam}(f^i(V)) < \xi$. This implies by Lemma 3.2 that the criticality of $f^n|_V$ is at most $d + r$. Then by Lemma 3.1 and by the choice of δ it follows that we must have $\operatorname{diam}(V') < \varepsilon$, a contradiction with the assumption. Hence there exists i, $0 \leqslant i \leqslant n - 1$ such that $\operatorname{diam}(f^i(V)) \geqslant \xi$. $\square$

Given a square S and a square S^l with $l > 1$ call any square S' contained in $S^l \setminus S$ and having a side intersecting with S and a side intersecting with S^l a *collar* square of S, S^l (references to one or even both squares S, S^l may be omitted if this does not cause any confusion). There is a well-defined function $\eta(l) = 4l/(l - 1) > 1$ such that the entire "collar" $S^l \setminus S$ is covered by $\eta(l)$ collar squares. Lemma 3.4 shows that under the assumptions of bounded recurrent criticality the fact that a pull-back of a square S is big implies that a certain collar square to S has a relatively big pull-back too. As we shall see later, together with Lemma 3.3 this implies that yet another square, though smaller than S, has a uniformly bounded away from 0 diameter of one of its pull-backs which eventually leads to a contradiction. Of the lemmas proven so far Lemma 3.4 seems to be the most important.

Lemma 3.4. *Let numbers $l > 1$, $\varepsilon < \xi/2, \gamma, r$ be given. Then there exists $\delta' = \delta'(\varepsilon, l, \gamma, r)$ such that the following holds. Let S be a square of radius less than δ' such that $d(S^l, p) > \gamma$ for any parabolic or attracting periodic point p and let V be a n-pull-back of S^l with recurrent criticality r. Suppose that there exists an n-pull-back of S corresponding to V and such that its diameter is greater than ε. Then there exists a collar square of S, S^l with a pull-back corresponding to V of diameter at least $\varepsilon/(2d^{2d+r}(\eta(l) + 1)) = \varepsilon'$.*

Proof. As δ' we choose the number $\delta' = \delta(\varepsilon', 1/l, \gamma, 2d + r)$ defined in Lemma 3.1. Thus, if the distance of a square $\tilde{S}$ from parabolic and attracting points is at least γ and its k-pull-back $\tilde{V}$ is such that the criticality of $f^k|_{\tilde{V}}$ is at most $2d + r$, then all pull-backs of $\tilde{S}^{1/l}$ corresponding to V have diameters less than ε' (all that follows from Lemma 3.1).

Suppose that all collar squares of S, S^l are such that all their pull-backs corresponding to V have diameters less than ε'. Choose $\eta(l)$ collar squares of S, S^l. Consider for every $i \geqslant 0$ the set $f^{-n+i}(S) \cap f^i(V)$, that is the full preimage of S inside the appropriate pull-back of S^l. By the assumption, V contains a pull-back of S with diameter greater than ε, so $f^{-n}(S) \cap V$ has diameter greater than ε. Thus we can find the greatest i for which $\operatorname{diam}(f^{-n+i}(S) \cap f^i(V)) \geqslant \varepsilon$. Then for all j such that $n \geqslant j > i$ we have $\operatorname{diam}(f^{-n+j}(S) \cap f^j(V)) < \varepsilon$.

Let us show that then $\operatorname{diam}(f^j(V)) < \varepsilon + 2\varepsilon'$ for each j such that $i < j \leqslant n$. For any point $x \in f^j(V)$ choose a point $x' \in f^{-n+j}(S) \cap f^j(V)$ so that x, x' belong to the same pull-back of a collar square of S, S^l (x' will have to be chosen on the boundary of such pull-back). Let us explain how we choose x' in more detail. If $x \in f^{-n+j}(S) \cap f^j(V)$, then we choose $x' = x$. If not, then x is the f^{n-j}-preimage of a point from a collar square. Choose the corresponding pull-back of this collar square and a point on the intersection of the boundary of this collar square with S which we also pull back to some point x'.

Since by the construction x, x' belong to the same pull-back of a collar square, the distance between x, x' is less than ε'. In other words, any point $x \in f^j(V)$ can be approximated by a point $x' \in f^{-n+j}(S) \cap f^j(V)$ such that the distance between x, x' is less than ε'. On the other hand, the diameter of $f^{-n+j}(S) \cap f^j(V)$ is less than ε by the assumption. The triangle inequality implies that then $\operatorname{diam}(f^j(V)) < \varepsilon + 2\varepsilon'$ for each j such that $i < j \leqslant n$. Since $\varepsilon + 2\varepsilon' < 2\varepsilon < \xi$, we see that $\operatorname{diam}(f^j(V)) < \xi$ for any $j = i+1, \dots, n-1$.

Hence by Lemma 3.2 the criticality of $f^{n-i-1}|_{f^{i+1}(V)}$ is at most $d + r$, and so the criticality of $f^{n-i}|_{f^i(V)}$ is at most $2d + r$. Since $\operatorname{diam}(S^l) < \delta' = \delta(\varepsilon', 1/l, \gamma, 2d + r)$ we conclude that every $(n-i)$-pull-back of S corresponding to $f^i(V)$ has diameter less than ε'. Now, since the criticality of $f^{n-i}|_{f^i(V)}$ is at most $2d + r$, we see that a collar square of S, S^l has at most $M = d^{2d+r}$ $(n-i)$-pull-backs corresponding to $f^i(V)$. On the other hand, we can cover $S^l \setminus S$ with $\eta(l)$ collar squares. Hence, there are no more than $M\eta(l)$ of $(n-i)$-pull-backs of collar squares of S, S^l corresponding to $f^i(V)$ (and hence contained in $f^i(V)$).

Thus, the set $f^i(V)$ is the union of no more than M pull-backs of S, each of which has diameter less than ε' by the choice of δ', and $M\eta(l)$ pull-backs of the originally chosen $\eta(l)$ collar squares with each pull-back being of diameter less than ε' by the assumption. Then the diameter of their connected union V is less than or equal to the sum of all these diameters which is at most $M\varepsilon' + \eta(l)M\varepsilon' = M(\eta(l) + 1)\varepsilon' < \varepsilon$. However the assumption is that the full preimage of S inside $f^i(V)$—and therefore $f^i(V)$ as a whole—have diameter at least ε, a contradiction. □

We are ready now to prove the main result of this section. As one can see, Theorem 3.5 almost literally repeats Lemma 3.1 with one main exception: criticality is replaced by recurrent criticality.

Theorem 3.5. *Suppose that $\varepsilon > 0$, $0 < k < 1$, $\gamma > 0$, r are given. Then there exists $\beta = \beta(\varepsilon, k, \gamma, r)$ such that the following holds. Let S be a square of radius less than β such that $d(S, p) > \gamma$ for any parabolic or attracting periodic point p. Suppose that V is an n-pull-back of S such that $f^n|_V$ has recurrent criticality at most r. Then any n-pull-back W of S^k corresponding to V has diameter at most ε and is such that the criticality of $f^n|_W$ is at most $d + r$.*

Proof. Clearly we may assume that $\varepsilon < \xi/10$. Suppose that for some square S, whose distance from parabolic and attracting periodic points is at least γ, there exists an n_0-pull-back U_0 of S^k of diameter less than ε, while the recurrent criticality of the corresponding to U_0 n_0-pull-back of S is at most r. Denote by V the n_0-pull-back of S corresponding to U_0. Our aim is to construct a sequence of squares $S_0, S_1, \dots$ which will have bounded away from 0 diameters of their appropriately chosen pull-backs under bounded from above by n_0 iterates of f. On the other hand the squares $S_0, S_1, \dots$ will have specific sizes converging to 0 and simultaneously guaranteeing that the squares will stay inside a fixed square $\hat{S}$ concentric with S and such that $S^k \subset \hat{S} \subset S$. The latter tells us that the squares $S_0, S_1, \dots$ will all be no closer than γ to parabolic/attracting points and also that these squares will be contained well inside S on which the n_0-pull-back corresponding to U_0 has recurrent criticality at most r. Therefore, technical lemmas proven above will be applicable provided the original square S is chosen to be small enough. Clearly, this picture eventually leads to a contradiction.

The actual construction relies upon the choice of a convenient parameter $l > 1$ such that

$$1 < l < \frac{2k+2}{3k+1}$$

(observe that $0 < k < 1$ implies that

$$1 < \frac{2k+2}{3k+1} < 2$$

and so such number l exists and is always less than 2). For the sake of computations made in this paragraph only we assume that the radius of S is 1 and set $S_0 = S^k$, so the radius of S_0 is k. Then the construction will be such that on each step the square S_{i+1} will be non-disjoint from S_i and will have the radius equal $l-1$ times the radius of S_i. Hence the radius of S_j is $k(l-1)^j$ and the radii of squares S_j form a decreasing geometric progression (recall that $l < 2$). To estimate how far from the center of S these squares can reach we need to sum up the series

$$k\left(1 + 2\sum_{j=1}^{\infty}(l-1)^j\right) = \frac{lk}{2-l}.$$

It is easy to see that our choice of l guarantees that

$$\frac{lk}{2-l} < \frac{k+1}{2} = t < 1.$$

Therefore, any sequence of squares described above will stay inside the square $\hat{S} = S^t$ and so for any j the distance between S_j and any parabolic or attracting point is less than γ, which makes previously proven lemmas applicable to squares S_j provided the size of the original square S is appropriately small. Observe that if S_j, $j \geqslant 1$, is one of our squares, then the square S_j^2 is contained in S.

Now that the number l has been chosen we can choose β. To do so set $l' = (l+1)/2$ and then apply Lemma 3.4 and choose the number $\delta' = \delta'(\varepsilon, l', \gamma, r)$. Thus, if there is a square R such that $\operatorname{diam}(R) \leqslant \delta'$, $R^{l'} \subset S$ and there is a pull-back of R corresponding to

V and of diameter greater than ε, then there is a collar square of R, $R^{l'}$ whose appropriate pull-back is of diameter at least

$$\frac{\varepsilon}{2d^{2d+r}(\eta(l')+1)} = \varepsilon'.$$

Then we apply Lemma 3.1 and find the number $\delta = \delta(\varepsilon', 1/2, \gamma, r+d)$. By Lemma 3.3, δ has the following property: if a square $R \subset S$ is such that $R^2 \subset S$ and also R has a pull-back corresponding to V and such that the diameter of this pull-back is at least ε', then there exists a pull-back of R^2 corresponding to V and of diameter greater than ξ. Now, as our number β we choose the smaller of δ, δ'; this ensures that both Lemmas 3.3 and 3.4 will be applicable throughout the argument.

Let us now prove that β has the properties from the lemma. Assume that this fails. Then for some square S of radius less than β, whose distance from parabolic and attracting periodic points is at least γ, there exists an n_0-pull-back U_0 of S^k of diameter at least ε while the recurrent criticality of the corresponding to U_0 n_0-pull-back of S is at most r. Denote by V the n_0-pull-back of S corresponding to U_0. To show that this leads to a contradiction we construct a sequence of squares following the ideas described above.

The first square in the sequence is $S^k = S_0$. To choose S_1 we proceed as follows. Our assumption is that U_0 is an n_0-pull-back of S_0 of diameter at least ε corresponding to an n_0-pull-back V of S such that the recurrent criticality of $f^{n_0}|V$ is at most r. Then by Lemma 3.4 and by the choice of β we can choose a collar square S' of S_0, $S_0^{l'}$ such that there exists a pull-back V' of S' corresponding to V with

$$\operatorname{diam}(V') \geqslant \frac{\varepsilon}{2d^{2d+r}(\eta(l')+1)} = \varepsilon'.$$

Because of the choice of β we then can apply Lemma 3.3 to the square $S'^2 = S_1 \subset S$ and conclude that it has a pull-back U_1 corresponding to V which has diameter at least $\xi > \varepsilon$. Observe that the radius of the square S_1 is $l-1$ times the radius of the square S_0 (this is exactly why we needed one more constant l'). We can assume that U_1 is an n_1-pull-back of S_1 corresponding to V and then $n_1 \leqslant n_0$. By the arguments from the second paragraph of this proof, $S_1 \subset S^t \subset S$ and so the distance between S_1 and any parabolic or attracting point of f is at least γ and we can repeat the argument.

Literally the same arguments apply to the square S_i which will be constructed after i steps in the process described above. That is, by the construction we will know that S_i has an n_i-pull-back of diameter at least ε; moreover, all squares S_j are consecutively non-disjoint and of radii $k(l-1)^j$ respectively, and $n_0 \geqslant n_1 \geqslant \cdots \geqslant n_i$. Then we will apply Lemma 3.4 and find a collar square R of S_i, $S_i^{l'}$ whose appropriate n_i-pull-back is of diameter greater than ε'. Then we will apply Lemma 3.3 and show that the square R^2 has an appropriate n_{i+1}-pull-back of diameter a least ξ, and hence of diameter greater than ε. Moreover, we will have $n_{i+1} \leqslant n_i$. Hence the construction can be repeated infinitely many times. However, then the radius of S_j converges to 0 and the iterates n_j of f for the pull-backs U_j of diameter greater than ε stay less than n_0, which is clearly impossible. This contradiction proves the first part of the claim of the theorem; the second part dealing with criticality follows immediately from the first one and Lemma 3.2. $\square$

As an application of Theorem 3.5 consider a point x which does not belong to the union P_r of the limit sets of recurrent critical points of f and to the set of parabolic and attracting periodic points of f. Then if δ is small enough, the square S of radius δ centered at x is disjoint from P_r and its distance from the set of parabolic and attracting periodic points of f is at least γ. Hence recurrent criticality of any pull-back of S is zero and moreover Theorem 3.5 applies. By this theorem, for a given $0 < k < 1$ we can find β such that any square S^k has only pull-backs of diameter at most ε, and so at the point x the map f is backward stable. Thus, we obtain another proof of one of the results of [19].

4. Dendrites

This section is devoted to studying pull-backs of dendrites under rational and polynomial maps. The main problem with using dynamical properties of maps of dendrites in our circumstances is that because we consider all points on the plane we need to consider their orbits and pull-backs under the plane map in question and not just under the restriction of this map onto the dendrite. Therefore, if U is an open connected set such that $U \cap D$ is connected (here D is a dendrite), then a decent pull-back of U in the sense of a rational map may have a disconnected intersection with the dendrite and thus may well correspond to two or more pull-backs of $U \cap D$ taken in the one-dimensional sense. Clearly, studying of possible cases when this phenomenon takes place is rather important for us, and if we can exclude it one way or another this would allow us to proceed with the tools developed in [10]. In this investigation we will need some easy topological properties of dendrites; in particular, it is well-known (see, e.g., [11]) that dendrites are uniquely arcwise connected, which will be used later on.

Some tools which allow us to fight this problem were suggested in [10]. More precisely, what was done in this direction in [10] is that in the case of a forward invariant graph G one can extend it and construct a larger forward invariant graph G' such that for G' the breakdown of connectivity described in the preceding paragraph can only happen after at least one critical point was hit by the pull-backs of U. This was enough in [10] because we only needed to pull back neighborhoods until a critical point is hit for the first time. However this is not enough in the present paper, so another set of tools is needed.

Lemma 4.1. *The image of a Jordan curve contained in $\mathbb{C}$ under a polynomial cannot be simply connected.*

Proof. Let f be a polynomial, let H be a Jordan curve bounding a set G, and suppose that $f(H)$ is simply connected. Then the boundary of $f(G)$ is contained in $f(H)$, but $f(G)$ is bounded and has a non-empty interior, a contradiction. □

Lemma 4.2. *Let f be a polynomial. Suppose that $U \subset \mathbb{C}$ is a closed Jordan disk and D is a dendrite such that $D \cap U$ is connected. If the boundary of U contains no critical values of f and V is a 1-pull-back of U such that $D \cap U$ contains images of all critical points of $f|_V$, then V is a closed Jordan disk and $f^{-1}(D) \cap V$ is a dendrite.*

Proof. We can cover the intersection of D and the boundary of U by finitely many small Jordan disks $U_1, \dots, U_k$ and find a subdendrite $D' \subset D$, contained in the interior of U, and intersecting each $U_i \cap D$. Since the boundary of U contains no critical values of f, if U_i's are sufficiently small, then each component of $f^{-1}(U_i)$ is mapped by f homeomorphically onto U_i. Therefore, if we prove that $f^{-1}(D') \cap V$ is connected, it will follow that $f^{-1}(D) \cap V$ is also connected. Then, since f is a polynomial (so it has nice local structure), the only reason why $f^{-1}(D) \cap V$ would not be a dendrite could be that it contained a loop. However, this is impossible by Lemma 4.1.

Thus, it remains to prove that $f^{-1}(D') \cap V$ is connected and V is a closed Jordan disk. Clearly, we may assume that $D' \cap U$ contains images of all critical points of $f|_V$. Therefore $U \setminus D'$ is topologically an annulus and f maps $V \setminus f^{-1}(D')$ onto it as a local homeomorphisms. By the Riemann–Hurwitz formula, the set $V \setminus f^{-1}(D')$ has Euler characteristic zero, so it is also topologically an annulus. Therefore $f^{-1}(D') \cap V$ is connected and V is a closed Jordan disk. $\square$

Lemmas 4.1 and 4.2 are useful tools in studying pull-backs of dendrites under rational maps. In the next lemma speaking of distances between points we use the standard spherical metric on S^2. By $P_r(f)$ we denote the union of limit sets of recurrent critical points of f, and by N the north pole of the sphere.

Lemma 4.3. *Let f be a dendrite-critical polynomial and D be the dendrite from its definition. Then there exists a forward invariant dendrite $D' \subset D$ with respect to which f is dendrite-critical, and a number $\varepsilon > 0$, for which the following holds: whenever W is a closed Jordan disk W such that $\operatorname{diam}(W) < \varepsilon$, the boundary of W is disjoint from the critical orbits of f, W intersects $P_r(f)$ and $W \cap D'$ is connected, then for every pull-back V of W the set $V \cap D'$ is also connected.*

Proof. Recall that we consider f as a map on the sphere, and so the case when D contains N will have to be considered. In fact, this case is a bit harder to tackle, so to begin with we assume that D does not contain N. We show that then we can set $D' = D$.

Suppose that U is a closed Jordan disk such that U does not contain N, the boundary of U is disjoint from critical orbits of f, and $U \cap D$ is connected. Let us prove that then for any 1-pull-back V of U the set $V \cap D$ is connected and V is a closed Jordan disk. Choosing as ε the distance between D and N and applying this claim inductively to a closed Jordan disk W with properties from the lemma, we will complete the proof.

Suppose that $V \cap D$ is not connected. Construct a new dendrite T in such a way that $T \supset D$, T contains all critical values of f and $T \cap U$ is connected. Then by Lemma 4.2 $f^{-1}(T \cap U) \cap V$ is a dendrite and V is a closed Jordan disk. Since by the assumption $f^{-1}(D) \cap V$ is not connected then there exist at least two distinct components of $f^{-1}(D) \cap V$. We can find an arc $I \subset f^{-1}(T \cap U) \cap V$ connecting them so that all points of I (except for the endpoints a, b) do not belong to D. Connect a and b with the arc J inside D. Then $I \cup J = H$ is a Jordan curve. Observe that by the construction and the assumptions H does not contain N while the image of H is contained in T and is therefore simply connected. This contradicts Lemma 4.1 and shows that $f^{-1}(D) \cap V$ is connected. Observe that here the size of U does not matter.

It remains to consider the case when $N \in D$. Then N is a cut point of D. The components of $D \setminus N$ are mapped one onto another in a well-defined fashion because N is fully invariant. Choose a closed Jordan disk U containing N in its interior and show that then there are only finitely many components of $D \setminus N$ not contained in U. Indeed, otherwise we can choose a sequence of such components $B_1, B_2, \dots$ so that there are points $x_i \in B_i \cap \partial U$ with $x_i \to x \in \partial U$. Then small neighborhoods W of x are such that $W \cap D$ is disconnected (because it has points of all B_i with big i and B_i are distinct components of $D \setminus N$), a contradiction to the local connectivity of D. Hence for every U there are only finitely many B_i's not contained in U.

Hence there are finitely many components of $D \setminus N$ intersecting $P_r(f)$. Denote these components of $D \setminus N$ by $A_1, \dots, A_n$. Let us show that the map f permutes them. Indeed, given A_l there exists a point $x \in \omega(c) \cap A_l$ where $c \in D$ is a recurrent critical point. Then there exists a point $y \in \omega(c)$ such that $f(y) = x$. Choose r so that $y \in A_r$. Then clearly $f(A_r) \subset A_l$. In other words, f maps A_i's so that the corresponding map ϕ of the finite set $\{1, \dots, n\}$ is surjective. Hence ϕ is a permutation as desired. Set $D' = N \cup \bigcup_{i=1}^n A_i$. Clearly, f is dendrite-critical with respect to D'. Choose $\delta > 0$ smaller than the distance between N and the filled-in Julia set $K(f)$. Then for every pair of subscripts i, j the distance between the sets $A_i \setminus B(N, \delta)$ and A_j is positive. Choose $\varepsilon < \delta$ which is less than the minimal distance between any two such sets.

Assume that W is a closed Jordan disk of diameter less than ε, intersecting a set A_i (e.g., this is so if W intersects $P_r(f)$). Observe that then i is unique and W does not contain N. Suppose that R is a k-pull-back of W intersecting D'. Then $R \cap D' = R \cap A_j$ where the set A_j is a unique set from the collection $A_1, \dots, A_n$ with $f^k(A_j) \subset A_i$. Suppose that for a 1-pull-back V of W the intersection $V \cap A_j$ is not connected. Create a new dendrite $T \supset (D' \cap W)$ so that T contains all critical values of f in W and $T \cap W$ is connected. By Lemma 4.2 the set $f^{-1}(T) \cap V$ is a dendrite. Since by the assumption $V \cap D' = V \cap A_j$ is not connected, we can find an arc I with the endpoints $a \in A_j$, $b \in A_j$ such that the image of I is contained in T and I is disjoint from D' except at a, b. Also, connect a, b with the arc J inside A_j. Then J is contained in A_j and avoids N. The arc I avoids N by the construction. Thus, the Jordan curve $I \cup J$ avoids N while its image is contained in the dendrite T, a contradiction with Lemma 4.2. This completes the proof of the lemma. □

Finally we prove the following lemmas in which the standard interval notation is adopted.

Lemma 4.4. *Suppose that f is a rational map with an invariant dendrite G. Let $I \subset G$ be a connected set such that $I \cap f(I) \neq \emptyset$ and*

$$\liminf_{n\to\infty} \operatorname{diam}\bigl(f^n(I)\bigr) = 0.$$

Then the set $K_0^\infty = \bigcup_{j=0}^\infty f^j(I)$ is connected and the following possibilities hold:

(1) *the set K_0^∞ contains a neutral fixed point*;
(2) *the orbit of I converges to an attracting fixed point*;
(3) *the orbit of I converges to a parabolic fixed point a so that for some point $d \in G$ the component U of $G \setminus \{d, a\}$ containing (d, a) is such that $f(U) \subset U$, all points of U*

are attracted to a and I is mapped inside U by some iterate of f; moreover, the point d can be chosen arbitrarily close to a.

Thus, if K_0^∞ does not contain a neutral fixed point, then $\lim_{n\to\infty} \operatorname{diam}(f^n(I)) = 0$.

Proof. Since I intersects $f(I)$, then also $f^n(I)$ intersects $f^{n+1}(I)$ for each n. Therefore the sets $K_m^n = \bigcup_{j=m}^n f^j(I)$ (including the case $n = \infty$) are connected. Since $\liminf_{n\to\infty} \operatorname{diam}(f^n(I)) = 0$, there is a sequence $k_n \to \infty$ with $f^{k_n}(I) \to a$, $a \in G$. Then $f^{k_n+1}(I) \to f(a)$, and since $\operatorname{diam}(f^{k_n}(I) \cup f^{k_n+1}(I)) \to 0$ we get $f(a) = a$.

Let us show that a must be attracting or neutral. Indeed, suppose that a is repelling. Then for any point x closer than some $\delta > 0$ to a we have $d(f(x), a) > d(x, a)$. Now, consider two cases. First assume that $a \in I$. To see that

$$\liminf_{n\to\infty} \operatorname{diam}\bigl(f^n(I)\bigr) = 0$$

is impossible, choose small $\varepsilon > 0$ so that $\varepsilon < \delta$ and I is not contained in the ε-ball B centered at a. By connectivity there exists a point $y \in I$ such that $d(y, a) = \varepsilon$. Thus $d(f(y), a) > \varepsilon$ and $f(I)$ is not contained in B either. By induction it implies that $\operatorname{diam}(f^k(I)) > \varepsilon$ for every k, a contradiction. Now, consider the case when $a \notin I$. Choose $\varepsilon < \delta$ so small that the ε-ball B centered at a is disjoint from I. Let us show by induction that no image of I is contained in B. Indeed, this is true for I. If it fails, it has to fail for the first time for some n. Then $f^{n-1}(I)$ is partially outside B while on the other hand it intersects $f^n(I) \subset B$. Hence by connectivity there is a point $y \in f^{n-1}(I)$ whose distance from a is ε. This implies that $d(f(y), a) > \varepsilon$ while on the other hand $f(y) \in f^n(I) \subset B$, a contradiction. So in any case there exists a ball B centered at a such that $f^n(I) \not\subset B$ for any n. Clearly, it contradicts the fact that $f^{k_n}(I) \to a$.

If an image of I contains a neutral fixed point then (1) holds and there is nothing to prove. If a is an attracting fixed point then the fact that $f^{k_n}(I) \to 0$ implies that the orbit of I converges to this fixed point as desired and (2) holds. So from now on we may assume that neither (1) nor (2) takes place. That is, a is a neutral fixed point and the set K_0^∞ does not contain a. We may also assume that I is closed.

Since G is a dendrite, the set K_0^∞ is contained in a unique component A of $G \setminus \{a\}$. Consider the unique arc $J = [x, a) \subset \bar{A}$ such that $J \cap I = \{x\}$ (recall that we use usual interval notation here). Then $J \subset K_0^\infty$ because of the properties of dendrites. Each point $z \in J$ defines an open connected set $U_z \subset G$ which is a component of $G \setminus \{z, a\}$ containing (z, a). Since a is neutral we may assume that the point z is chosen so that $\bar{U}_z$ contains no critical points of f and the only fixed point of f in $\bar{U}_z$ is a. Since J is contained in K_0^∞, we have $f(z) \in K_0^\infty \subset A$. Consider two cases.

(i) $f(z) \notin \bar{U}_z$. Then the choice of z guarantees that $f([z, a]) \supset [z, a]$ and that points of J which are mapped by f back into J, are mapped farther away from a. Then the argument mimicking the previous argument dealing with repelling periodic points shows that in this case we have a contradiction. Indeed, take the first m such that $f^m(I) \subset U_z$. Since $f^{m-1}(I)$ intersects $f^m(I)$, we see that $f^{m-1}(I)$ intersects U_z. By the choice of m we see that $f^{m-1}(I)$ is not contained in U_z. Therefore by connectivity $z \in f^{m-1}(I)$ and so $f(z) \in f^m(I) \subset U_z$, a contradiction.

(ii) $f(z) \in \bar{U}_z$. By the choice of z we know that $f(z) \neq z$. Then $f(z) \in U_z$. Consider $f([z,a]) \cap [z,a]$; clearly, this is an interval of the form $[b,a]$ where $b \in [z,a]$ (we rely upon the fact that a is an endpoint of $\bar{A}$). Consider the point $d \in [z,a]$ such that $f(d) = b$. Then there are two possibilities. First, d may belong to (b,a). In this case the situation is like the one in (i), which leads to the contradiction.

Second, d may belong to $[z,b)$. In this case points on the interval $[d,a]$ are pushed closer to a on the same interval. We will show that then $f(U_d) \subset U_b$. Indeed, the map on $\bar{U}_d$ is injective. Therefore, the only point of $\bar{U}_d$ mapped by f to b is d. Given any other point $u \in U_d$, connect it by the unique arc $[u,v]$ to $[d,a]$ (so that $v \in (d,a)$). Then the image of $[u,v]$ is the arc connecting $f(v) \in (b,a)$ and $f(u)$. If $f(u) \notin U_b$ then there exists a point $t \in [u,v]$ mapped to b or to a, a contradiction with injectivity of $f|_{\bar{U}_d}$. This actually implies that the orbits of all points of U_d converge to a. Indeed, by the above argument $f(U_d) \subset U_b$, where $b = f(d)$. This can be repeated, which by induction implies that $f^n(U_d) \subset U_{f^n(d)}$. Since there are no other fixed points in $[z,a)$, we conclude that U_b can serve as the set U from the case (3) of the lemma.

Observe that if neither (1) nor (2) holds, then Snail lemma easily implies that a is parabolic. □

An easy analysis of the result proven in Lemma 4.4 using information about rational maps and their parabolic points leads to the following corollary given here without proof. To state it denote by A the component of $G \setminus \{a\}$ containing U.

Corollary 4.5. *In the situation of Lemma* 4.4(3) *the point d can be chosen in such a way that $\bar{U} \cap J(f) = \{a\}$. Moreover, the distance between U and the set $A \cap J(f)$ is then positive.*

We are now ready to prove Lemma 4.6 which will be applied in the next section.

Lemma 4.6. *Suppose that f is a rational map with an invariant dendrite D. Suppose that $[a_n, b_n] = I_n$, $i = 1, 2, \dots$, is a sequence of arcs in D converging to an arc $I' = [a,b]$ and such that $a_n, b_n \in J(f)$. Moreover, suppose that there exists a sequence $(m_n)_{n=1}^{\infty}$ such that* $\operatorname{diam}(f^{m_n}(I_n)) \to 0$ *and $f^{m_n}(I_n)$ does not contain neutral periodic points for any n. Then $I = (a,b)$ is wandering, and so* $\operatorname{diam}(f^k(I')) \to 0$ *and $\omega(a) = \omega(b)$.*

Proof. We may assume that $a \neq b$ (therefore the images of I are never degenerate) and $m_n \to \infty$. Observe that $a, b \in J(f)$. Assume that $I = (a,b)$ is not wandering. Then there exist two positive integers k, l such that $f^k(I) \cap f^{k+l}(I) \neq \emptyset$, and so there are points $x, y \in I$ such that $f^k(x) = f^{k+l}(y)$. For the sake of simplicity we assume that $k = 0$ and $l = 1$; the same arguments can be repeated in general with the appropriate changes (e.g., one will have to consider a certain iterate of f and not f itself, etc.).

By the assumptions, every compact subinterval K of I is contained in all but finitely many I_n's and if K is big enough then it contains both x and y so that $K \cap f(K) \neq \emptyset$. Hence Lemma 4.4 and Corollary 4.5 apply to K. Observe that the images of K cannot contain neutral periodic points because of the assumptions on I_n. Thus we need to consider cases covered by Lemma 4.2(2) and (3).

First consider a simpler case, covered in Lemma 4.4(2), when the orbit of K converges to an attracting fixed point. Each I_n contains points of $J(f)$, and so does $f^{m_n}(I_n)$. On the other hand, if n is big then $f^{m_n}(I_n) \supset f^{m_n}(K)$ and the set $f^{m_n}(K)$ gets closer and closer to the attracting fixed point a as n grows to infinity. Thus, if $\varepsilon > 0$ is less than one half of the distance between a and $J(f)$ then from some time on $\operatorname{diam}(f^{m_n}(I_n)) > \varepsilon$, a contradiction.

Suppose now that the parabolic case covered in Lemma 4.4(3) takes place. Fix a number N such that $f^N(K) \subset U$ where U is chosen as in Lemma 4.4(3) and Corollary 4.5. Clearly, for all big n we may assume that $m_n > N$ and $I_n \supset K$. Hence $f^{m_n}(I_n)$ is a set which on the one hand contains points of U and on the other hand contains points of $J(f)$. Moreover, the fact that $f^{m_n}(I_n)$ does not contain neutral periodic points implies that $f^{m_n}(I_n)$ is contained in the same component of $D \setminus \{a\}$ as U. Therefore by Corollary 4.5 we see that $\operatorname{diam}(f^{m_n}(I_n)) > \varepsilon$ for some $\varepsilon > 0$ and all sufficiently large n, a contradiction. □

5. Main results

The main aim of this section is to prove Theorem 5.4 and its corollary. First let us introduce some terminology assuming that a rational function f is given. For $x \in \hat{\mathbb{C}}$ and $n > 0$ consider the supremum $r_n(x)$ of all r such that $B(f^n(x), r)$ can be pulled back to x univalently. Then $r_n(x) > 0$ if all points $x, \dots, f^{n-1}(x)$ are not critical; otherwise define $r_n(x)$ as 0. Denote by V the pull-back of $B(f^n(x), r_n)$ corresponding to x; then there exists a critical point c_n belonging to the boundary of $f^{m_n}(V)$. We call (c_n, m_n) a *generating pair* for $r_n(x)$.

If $x \in J(f)$ and $r_n(x) \nrightarrow 0$, then x is called (C-)*reluctant* (recall that C denotes the set of critical points of f). The set of all such points is denoted by $\mathrm{Rlc}(f)$ (reluctant points are also called *conical*, see, e.g. [12] and the set $\mathrm{Rlc}(f)$ is also called the *radial Julia set of* f, see [20]; in [16, Section 8.3] reluctant points are discussed in the context of Kleinian groups). If $x \in J(f)$ and $r_n(x) \to 0$, the point x is called (C-)*persistent*. There are trivial cases when a point x is persistent, e.g., if it is precritical, or preparabolic. Given a persistent point x let us call the sequence of generating pairs (c_n, m_n) with $n = 1, 2, \dots$ the *standard basic setup* for the point x. The set of all persistent points is denoted by $\mathrm{Prs}(f)$. By the definition, $\mathrm{Prs}(f) \subset J(f)$ and $\mathrm{Rlc}(f) \subset J(f)$. Finally, denote by $PA(f)$ the union of all parabolic periodic points of f and call periodic orbits *cycles*.

The next lemma is useful in the proof of Theorem 5.3; it is used in the proof of Theorem 1.1 and will be useful for us as well.

Lemma 5.1 [8]. *If $z \in \mathrm{Prs}(f)$ is neither precritical nor preparabolic, then $\omega(z) \subset P_r(f)$.*

We will also need the following well-known lemma from topological dynamics (for the proof see, e.g., [10]).

Lemma 5.2. *Let X be a compact metric space and $T : X \to X$ a continuous map. Let $x \in X$, $M > 0$, and let $K \subset \omega(x)$ be a compact set such that $T^M(W \cap \omega(x)) \subset K$ for some open set $W \supset K$. Then $\omega(x) = \bigcup_{i=0}^{M-1} T^i(K)$. In particular, if there are pairwise* *disjoint non-empty compact invariant sets* $A_1, \dots, A_j$ *whose union is* $\omega(x)$ *then* $j = 1$ *and* $A_1 = \omega(x)$. *Moreover, finite limit sets are cycles.*

Before passing on to Theorem 5.3, we would like to point out that even though the restrictions included in the definition of a dendrite-critical rational map do not seem to be too strong, still one can show that dendrite-critical rational maps cannot have Cremer points. Indeed, suppose that this is false. Then without loss of generality we may assume that there exists a fixed Cremer point a. It is known that then a belongs to the limit set of a recurrent critical point c [19]. By the assumptions on dendrite-critical maps this implies that $c \in D$, and hence $a \in D$. Choose a small neighborhood U of a so that $U \cap D$ is connected and $c \notin \bar{U}$. Local connectedness of D implies that there are only finitely many components of $D \setminus \{a\}$ which are not contained in U. Denote those of them which contain points of $\omega(c)$ in their intersections with U by $A_1, A_2, \dots, A_l$. Also, since D is invariant then the following holds: for every component K of $D \setminus \{a\}$ there exists a unique component L of $D \setminus \{a\}$ such that $f(K \cap U) \subset L$.

Now, let us apply f to $A_1 \cap U$. By the definition it either gets mapped into A_i for some A_i, or it gets mapped into a small component $K \subset U$ of $D \setminus \{a\}$. In the latter case we can apply f over and over until the component in question maps into some A_i for the first time. This must happen because c must eventually exit U (after all, $c \notin \bar{U}$ is recurrent). Hence in either case we will find the next A_i to which the same arguments can be applied. Clearly this eventually leads to some j such that a small arc $I = [a, b] \subset A_j \cap U$ eventually maps into A_j. Choosing an even smaller arc I' inside $A_j \cap U$ with an endpoint at a we see that this arc will map over itself or into itself under an appropriate finite iterate of f. By Snail lemma this is impossible for a Cremer point a, a contradiction.

We can now pass on to Theorem 5.3.

Theorem 5.3. *Suppose that* x *is a persistent point of a dendrite-critical polynomial* f. *Then at least one of the following holds*:

(a) x *is precritical, or*
(b) x *is preparabolic, or*
(c) *there exists a recurrent critical point* $c(x)$ *such that* $\omega(x) = \omega(c(x))$.

Proof. Assume that x is neither preparabolic nor precritical. Then by Lemma 5.1, $\omega(x) \subset P_r(f)$. Now, just like in the definition of a dendrite-critical rational map there are two main requirements on recurrent critical points of f, there are two cases we need to deal with here. The first case is easier so we begin with it. Namely, suppose that $\omega(x) \not\subset D$. Then the set $\omega(x) \setminus D$ is contained in the union of all limit sets of recurrent critical points contained in $S^2 \setminus D$. Consider the union B of all recurrent critical points $c \in S^2 \setminus D$ such that $\omega(c) \cap (\omega(x) \setminus D) \neq \emptyset$. By the definition of a dendrite-critical map, any limit set $\omega(c), c \in B$, is minimal, and since all these sets $\omega(c)$ intersect $\omega(x)$, then $B' := \bigcup_{c \in B} \omega(c) \subset \omega(x)$. On the other hand, since D is invariant and all sets $\omega(c)$ with $c \in B$ are minimal, $B \cap D = \emptyset$ implies that $B' \cap D = \emptyset$. Hence, $\omega(x) = B' \cup (D \cap \omega(x))$, with both sets in the union invariant and disjoint. Since $B' \neq \emptyset$, it follows from Lemma 5.2 that $\omega(x) = B'$, and by the same lemma $B' = \omega(c)$ for some recurrent critical point $c \in B$ which completes the case when $\omega(x) \not\subset D$.

Let us now assume that $\omega(x) \subset D$. Denote by A the union of the set of all critical points of f and the set of all its parabolic periodic points. We want to choose Basic Setup so that x is A-followed. Then by Theorem 2.1 we will have that $\omega(x) = \omega(c(x))$ where $c(x) \in A$. Moreover, some extra properties which we will establish guarantee that $c(x) \in \omega(x)$.

Loosely, the idea is as follows. First, choose for every point $f^n(x)$ in the orbit of x a certain neighborhood W_n of $f^n(x)$ and a certain k_n-pull-back V_n of W_n. The pull-backs will be such that $V_n \ni c_n$ where $c_n \in A$. Then we set $m_n = n - k_n$ and thus complete Basic Setup—that is, up to the choice of the crucial elements of the construction which are W_n, V_n and k_n and up to the proof that x is A-followed. Now, certain sets on this list are not difficult to come up with. Indeed, we have total control over neighborhoods W_n, so they can be chosen in such a way that their diameters converge to 0 (in addition we want them to have connected intersections with D). However, we do not have control over V_n, and this is when the tools developed in Sections 3 and 4 become helpful.

We may assume that D has the properties from Lemma 4.3 and for any sufficiently small ε and any Jordan disk W such that $W \cap D$ is connected and W contains points of the set $P_r(f)$, all pull-backs V of W intersect D over a connected set.

Let us pass to the precise construction. It is done in steps, so let us describe the m-th step assuming that $1/m < \varepsilon$ where ε is the constant found in Lemma 4.3. We can also assume that ε is less than the distance between the union of all attracting periodic points of f and the Julia set $J(f)$, and that ε is less that the distance between any limit set of a recurrent critical point c not belonging to D and D itself (recall that such limit sets are minimal and therefore disjoint from D). Set $\varepsilon_m = 1/m$ and cover D with a finite collection $\mathcal{U}$ of closed Jordan disks $U_1, \dots, U_k$ whose intersections with D are connected (this is possible since D is locally connected), whose boundaries are disjoint from the orbits of critical points, and whose diameters are less than ε_m. Then we choose a Lebesgue number δ' for this cover. On the other hand we choose the number $\beta = \beta(\varepsilon_m, 1/2, \gamma, 1)$ from Theorem 3.5, where γ is the minimal distance between any $\bar{U}_i$ disjoint from $PA(f)$, and $PA(f)$. By Theorem 3.5 if we have two squares T and T^2, the diameter of T^2 is less than β, $d(T^2, PA(f)) > \gamma$ and there is a pull-back V of T^2 such that among sets $V, f(V), \dots, T^2$ the only one containing a recurrent critical point is V, then any corresponding to V pull-back of T will have diameter at most ε_m. Finally, we set $\delta_m = \min(\delta'/10, \beta)$.

The above implies that given a point $y \in D$, whose distance from $PA(f)$ is at least γ, we can first find j such that the ball of radius δ' centered at y is contained in U_j, and then a square S of radius δ_m centered at y. If we consider a pull-back of U_j of recurrent criticality 1 then corresponding to it pull-back of S will be of diameter at most ε_m. This fact which follows from of Theorem 3.5 plays an important role in the construction below.

Choose N_m so big that $r_n(x) < \delta_m$ for every $n \geqslant N_m$. The m-th step in the construction will be valid for the numbers n such that $N_m \leqslant n < N_{m+1}$. Let us explain how we choose basic setup. Given $n \geqslant N_m$ we first measure the distance between $f^n(x)$ and the parabolic periodic points of f. If there exists a parabolic periodic point a such that $d(f^n(x), a) \leqslant \varepsilon_m$ then we set $c_n = a, m_n = n$. Suppose that the distance between $f^n(x)$ and the parabolic periodic points of f is greater than ε_m. Then any set U_j containing $f^n(x)$ has the closure disjoint from $PA(f)$ because the diameter of any U_j is less than ε_m. Choose a set $U_{i(n)} \ni f^n(x)$ in such a way that the ball of radius δ_m centered at $f^n(x)$ is contained in $U_{i(n)}$ (it is possible because δ' is a Lebesgue number of $\mathcal{U}$).

The distance of the set $U_{i(n)}$ from $PA(f)$ is at least γ, so it is disjoint from $PA(f)$. Pull back $U_{i(n)}$ until it hits a recurrent critical point c, or until it hits the critical point c' generating r_n, whichever comes first. Set $c_n = c$ or $c_n = c'$ respectively. This defines the number m_n and ultimately our basic setup.

What needs to be proven now is that x is A-followed and $c(x) \in \omega(x)$. To prove that x is A-followed we need to verify several properties listed in Section 2. First, observe that Basic Condition is satisfied because $\varepsilon_m \to 0$ and $d(f^n(x), f^{n-m_n}(c_n)) < \varepsilon_m$ if $N_m \leqslant n < N_{m+1}$, so $d(f^n(x), f^{n-m_n}(c_n)) \to 0$ as $n \to \infty$. The main part of the verification of the fact that x is A-followed is to check if for any limiting pair (x', c') we have $\omega(x') = \omega(c')$. Consider a sequence of n-pairs with $n \to \infty$ which converge to a pair of points (x', c'). That is, suppose that $(x', c') = \lim(f^{l_n}(x), f^{l_n-m_n}(c_n))$ along a sequence of numbers n and with $m_n \leqslant l_n \leqslant n$. To prove that $\omega(x') = \omega(c')$ we make use of the fact that D is a dendrite.

Suppose that there is a sequence of n-pairs $(f^{l_n}(x), f^{l_n-m_n}(c_n))$ converging to (x', c') and such that in all of them c_n is a critical point generating r_n. This means that in the construction the pull-backs of $U_{i(n)}$ do not hit *recurrent* critical points until they reach c_n. Therefore by Theorem 3.5 we have $d(f^{l_n}(x), f^{l_n-m_n}(c_n)) < \varepsilon_k = 1/k$ where k is such that $N_k \leqslant n < N_{k+1}$. On the other hand, $k \to \infty$ as $n \to \infty$. Therefore in this case $d(f^{l_n}(x), f^{l_n-m_n}(c_n)) \to 0$ as $n \to \infty$ and so $x' = c'$ and $\omega(c') = \omega(x')$ as desired. Observe that in this case we essentially prove that if there is a sequence of n-pairs $(f^{l_n}(x), f^{l_n-m_n}(c_n))$ converging to (x', c') and such that c_n is a critical point generating r_n then $x' = c'$.

Suppose that there is a sequence of n-pairs $(f^{l_n}(x), f^{l_n-m_n}(c_n))$ converging to (x', c') and such that in all of them c_n is a parabolic periodic point. Then by the construction in this case $n = m_n$ and hence $l_n = m_n = n$ which implies that we have n-pairs $(f^n(x), c_n)$ converging to (x', c'). Clearly, this implies that c' is a parabolic periodic point and that $x' = c'$.

From now on we may assume that in our sequence of n-pairs giving rise to the limiting pair (x', c') all n-pairs $(f^{l_n}(x), f^{l_n-m_n}(c_n))$ arise from the pull-backs of $U_{i(n)}$ hitting a recurrent critical point c_n for the first time. Unlike before, in this case we rely upon topological and dynamical properties of dendrites established in Section 4. Observe first that by the choice of ε the point c_n must belong to D. Also, by the choice of $U_{i(n)}$ we know that it has diameter less than ε and contains some points of limit sets of recurrent critical points of f. Therefore Lemma 4.3 applies to $U_{i(n)}$ and all pull-backs of $U_{i(n)}$ will have connected intersections with D. In particular, if W is the $(n - m_n)$-pull-back of $U_{i(n)}$ corresponding to c_n, then $W \cap D$ is connected. Therefore there exists a unique arc I_n connecting $f^{m_n}(x)$ and c_n inside $W \cap D$. Observe that I_n has the endpoints c_n and $f^{m_n}(x)$ which both belong to $J(f)$. Observe also that $f^{m_n}(I_n) \subset U_{i(n)}$ which implies that $f^{m_n}(I_n)$ contains no attracting or parabolic periodic points (otherwise the choice of points in the basic setup for $f^n(x)$ would have been different by the definition). Hence Lemma 4.6 applies to the just constructed sequence of arcs I_n (recall that f has no Cremer points). It implies that $\omega(x') = \omega(c')$ as desired.

Observe that along the way we establish one extra property of our construction. Namely, if c is a non-recurrent critical point which appears in the basic setup infinitely many times, then it follows from Theorem 3.5 that $d(f^{m_n}(x), c) \to 0$ for m_n's corresponding to $c_n = c$ in basic setup. Therefore $L_c = \{c\}$ for any such critical point c.

Let us use this to complete the proof. Indeed, as we have just shown the point x is followed by points of A. Hence by Theorem 2.1 there exists a point $c \in A$ such that $\omega(c) = \omega(x)$ with all the properties listed in Theorem 2.1. We assume that x is not preparabolic. Then c cannot be a parabolic periodic point because it is known that $\omega(x)$ can be a parabolic cycle if and only if x is preparabolic. Hence c is either recurrent or non-recurrent critical point. Suppose that c is a non-recurrent critical point. Since by Theorem 2.1 c appears in basic setup infinitely many times, we see by the preceding paragraph that $L_c = \{c\}$. If the bounded case takes place, then for some m we have $f^m(x) = c$, which is impossible since x is not precritical. Hence the unbounded case takes place. By Theorem 2.1 c is then recurrent, because it belongs to L_c, a contradiction with the assumption that c is non-recurrent. Thus c is recurrent and so $\omega(x) = \omega(c)$ for some recurrent critical point c. $\square$

Together with Lemma 5.1 this theorem immediately implies our main Theorem 5.4.

Theorem 5.4. *For a dendrite-critical polynomial f and a conformal measure μ at least one of the following holds*:

(1) *For μ-a.e. $x \in J(f)$, $\omega(x) = J(f)$.*
(2) *For μ-a.e. $x \in J(f)$, $\omega(x) = \omega(c(x))$ for some critical point $c(x)$ depending on x, and at least one of the following holds:*
    (a) *x is an eventual preimage of $c(x)$, or*
    (b) *x is preparabolic, or*
    (c) *$c(x)$ can be chosen to be recurrent.*

An immediate corollary is the following.

Corollary 5.5. *If f is a dendrite-critical polynomial and μ is a non-atomic conformal measure, then for μ-a.e. point x the set $\omega(x)$ contains a critical point.*

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